ON THE STABILITY OF ROUGH SYSTEMS*

(OB USTOICHIVOSTI GRUBYKH SISTEM)

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Rough systems are non-linear systems for which the problems of stability can be solved correctly by fairly simple approximate methods. The most interesting of such systems are those for which the problem of the stability of motion reduces to the consideration of linear equations with constant coefficients.

A sufficiently general method for the treatment of rough systems was given by me in my book [1], page 194. One should take into account that the functions f_{sr} considered there were assumed to be only bounded. They may, therefore, depend on t as well as on the variables $x_1, \ldots, x_n$.

1. Let us consider the system of differential equations

$$\frac{dx_1}{dt} = p_{s1} x_1 + \ldots + p_{sn} x_n \qquad (s = 1, \ldots, n) \tag{1}$$

on the assumption that all the coefficients p_{sr} have the form

$$p_{sr} = c_{sr} + \varepsilon f_{sr}$$

where ϵ is a real parameter, the c_{sr} are real constants, and the f_{sr} are bounded real functions of the variables t, and, possibly, of $x_1, \ldots, x_n$ in the region $t \geqslant t_0$, and $x_1^2 + \ldots + x_n^2 < A$.

The given equations (1) can be juxtaposed with the equations with constant coefficients c_{sr}

$$\frac{dx_s}{dt} = c_{s1} x_1 + \ldots + c_{sn} x_n \qquad (s = 1, \ldots, n) \tag{2}$$

Regarding the roots λ_s of the characteristic equation of the latter system

$$\| c_{sr} - \delta_{sr} \lambda \| = 0$$

we make the assumption that for arbitrary non-negative integers $m_1, \ldots, m_n$, whose sum is 2, the expression $m_1\lambda_1 + \ldots + m_n\lambda_n$ does not vanish. By

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* The work was published in a small number of copies in 1953.

this assumption and on the basis of a known theorem of Liapunov, the partial differential equation

$$\sum_s \frac{\partial V}{\partial x_s}(c_{s1}x_1 + \ldots + c_{sn}x_n) = -(x_1^2 + \ldots + x_n^2) \tag{3}$$

determines uniquely the quadratic form with constant coefficients, $a_{sr} = a_{rs}$,

$$V = \frac{1}{2}\sum_{s,r} a_{sr}x_s x_r$$

For numerically small enough ϵ, and for a small enough positive μ (less than 1) the quadratic form

$$-\frac{dV}{dt} - \mu(x_1^2 + \ldots + x_n^2) =$$

$$= (1-\mu)(x_1^2 + \ldots + x_n^2) - \varepsilon \sum_{s,r} \frac{\partial V}{\partial x_s} f_{sr} x_r = \sum_{s,r} h_{sr} x_s x_r \qquad (h_{sr} = h_{rs})$$

will be positive for arbitrary values of the variables, if A is small enough. The total time derivative in the last expression was taken with the aid of the given equations (1).

The asymptotic stability or instability of the undisturbed motion ($x_1 = 0, \ldots, x_n = 0$) of the equations (2) with the constant coefficients c_{rs} corresponds exactly with the asymptotic stability and instability of the given system (1). The quantities ϵ and A, for which there unconditionally exists such a correspondence, are determined in accordance with Sylvester's theorem from the n inequalities

$$\begin{vmatrix} h_{11}, \ldots, h_{1r} \\ h_{r1}, \ldots, h_{rr} \end{vmatrix} > 0 \qquad (r = 1, \ldots, n) \tag{4}$$

for small enough μ.

The bounds on A and ϵ determined by the last inequalities, and also the bounds for the functions ϵf_{sr} can, of course, be made more precise, if in the equations (3), one considers in place of the expression $-(x_1^2 + \ldots + x_n^2)$ some other negative-definite form U with real coefficients.

2. One can give an estimate of the bounds of the largest and smallest deviations of the disturbed variables.

For this purpose let us consider the extremal values of the derivative

$$V' = -(x_1^2 + \ldots + x_n^2) + \varepsilon \sum_{s,r} \frac{\partial V}{\partial x_s} f_{sr} x_r = \frac{1}{2}\sum_{s,r} b_{sr} x_s x_r$$

on the surface $V = c$. By Lagrange's method we have for the extremum the equations with the multiplier λ

$$\frac{\partial V'}{\partial x_s} = \lambda \frac{\partial V}{\partial x_s} \tag{5}$$

whence

$$\frac{1}{2}\sum_{\alpha,\beta}\frac{db_{\alpha\beta}}{\partial x_s}x_\alpha x_\beta + \sum_\alpha b_{s\alpha}x_\alpha = \lambda\sum a_{s\alpha}x_\alpha$$

Symmetrizing these equations, we obtain

$$\frac{1}{2}\cdot\frac{1}{2}\sum_{\alpha,\beta}\left(\frac{\partial b_{\alpha\beta}}{\partial x_s}x_\beta + \frac{\partial b_{\beta\alpha}}{\partial x_s}x_\alpha\right)x_\alpha + \sum b_{s\alpha}x_\alpha = \lambda\sum a_{s\alpha}x_\alpha$$

and, hence, find that λ must satisfy the equation

$$\left\|\frac{1}{4}\sum_\beta\left(\frac{\partial b_{r\beta}}{\partial x_s}x_\beta + \frac{\partial b_{\beta r}}{\partial x_s}x_\alpha\right) + b_{sr} - \lambda a_{sr}\right\| = 0$$

In the region $x_1^2 + \ldots + x_n^2 < A$, and for t in the interval (t_0, t), let λ_1 be the smallest and λ' be the largest root of this equation.

For the case of rough stability, when the function V is definitely positive, we obtain through multiplication of (5) by the variables x_s, and through addition,

$$\sum\frac{\partial V'}{\partial x_s}x_s = 2\lambda V, \quad \text{or} \quad \frac{1}{2}\sum_{s,\alpha,\beta}\frac{\partial b_{\alpha\beta}}{\partial x_s}x_s x_\alpha x_\beta + 2V' = 2\lambda V$$

On the surface $V = c$, and for small enough A, the following inequality is valid

$$\varepsilon_1 2V \leqslant -\frac{1}{2}\sum_{s,\alpha,\beta}\frac{\partial b_{\alpha\beta}}{\partial x_s}x_s x_\alpha x_\beta \leqslant \varepsilon' 2V$$

where ϵ_1 and ϵ' are small enough numbers.

Hence

$$(\lambda_1 + \varepsilon_1)V \leqslant V' \leqslant (\lambda' + \varepsilon')V$$

and, hence,

$$V_0 e^{(\lambda_1+\varepsilon_1)t} \leqslant V \leqslant V_0 e^{(\lambda'+\varepsilon')t}$$

If the initial disturbances $x_{10}, \ldots, x_{n0}$ were on the sphere $x_{10}^2 + \ldots + x_{n0}^2 = c$, then

$$\varkappa_1 c \leqslant V_0 \leqslant \varkappa_n c$$

where χ_1 and χ_n denote the largest and smallest root of the equation $||a_{sr} - \delta_{sr}\chi|| = 0$, respectively.

In the case of rough stability, when V is positive-definite, these roots are positive.

The points $(x_1, \ldots, x_n)$ in which Liapunov's function has the value V, are in the region

$$x_1^2 + \ldots + x_n^2 \leqslant \frac{V}{\varkappa_1}$$

From this it follows that the square of the radius of the sphere, in which the point in the disturbed motion will lie under the initial condition $x_{10}^2 + \ldots + x_{n0}^2 = c$, will satisfy the inequality

$$x_1^2 + \ldots + x_n^2 \leqslant c \frac{\varkappa_n}{\varkappa_1} e^{(\lambda' + \varepsilon')t}$$

It is obvious that the bounds given by the last inequalities for the deviation of disturbed variables x_s could be improved if in Equations (3) one would consider in place of the expression $x_1^2 + \ldots + x_n^2$ some other positive-definite quadratic form U with real constant coefficients.

BIBLIOGRAPHY

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